Int J Theor Phys (2010) 49: 936–942
DOI 10.1007/s10773-010-0272-y

Classical-quantum Interface of a Particle in a Time-dependent Linear Potential

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Received: 30 August 2009 / Accepted: 7 February 2010 / Published online: 24 February 2010

Abstract We use a wave packets approach to analyze the non-trivial time-dependence solution of quantum mechanical systems of a particle in a linear potential. We relate the system of a free particle with that of a particle in a time-dependent linear potential by the use of the de Broglie hypothesis which is one important aspect of the study of the classical-quantum interface. Closed-form analytic results have been obtained as: (i) nonspreading Airy wave packets, (ii) Gaussian wave packets.

Keywords Time-dependent Schrödinger equation · de Broglie hypothesis · Wave packets

There has been considerable interest in the system of a particle in a linear potential (with time -dependent parameters) where the exact propagator has long been known [1]. This model has eigenfunctions (wave packets) described by the Airy function [2]. This system and the Airy wave functions on a half-line have been used to model the production of high harmonic generation in the laser irradiation of rare gases [3, 4], and the edge electron gas [5–7]. The model on piecewise domains and the wave functions have been frequently used to model various physical systems [8, 9]. Recent work [10–22] focus further on exact solutions and their properties and several method have been used to solve this system.

Using a particular unitary transformation operator, which resembles the one responsible for the existence of coherent states in harmonic oscillators, Song [22] relates the system of a particle in a linear potential to that of a free particle and found a general wave packet described by an Airy function. The Schrödinger equation for a free particle has also long been interesting in that the equation is formally identical to the wave equation of a beam of light in the paraxial approximation [23–26]. In the present work, the relation between the two systems naturally arises. The unitary operator is not assumed beforehand, it will naturally appear in the derivation.

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In a previous recent work [27], we have derived the solutions of a quantum system subject to a spatially uniform time-dependent force and showed that the solutions, in discrete case, are generalized coherent states obtained by applying the displacement operator, $D(t)$, see [27] for details. The present work, while dealing with the same system, uses a different, original and elegant way to investigate the solutions.

The present approach is based on a particular realization of quantum mechanics of great practical importance, namely wave mechanics, used to describe the motion of a quantum particle in one-dimensional space. It is this realization which serves as the introduction to the fundamentals of quantum mechanics in most textbooks. We show that the construction of the solutions for a quantum system subject to a spatially uniform, time-dependent force is accomplished in such a way that the resulting states have similarity to the one for a quadratic system which give coherent states. The basic idea of these developments is to establish an intimate connection between the wave function of the solution of the Schrödinger equation for the free particle with the physical problem which is considered in the case of a particle in time-dependent linear potential. We use the de Broglie hypothesis "$\hbar k = p$", which is one important aspect of the study of the classical-quantum interface, to relate the simple model of a free particle to the model of a particle in a time-dependent linear potential. The obtained solutions are nothing but the (i) nonspreading Airy wave packets, (ii) Gaussian wave packets with a "center of mass" moving along the classical trajectory.

The problem is to find the solutions of the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = H\psi(x,t) \tag{1}$$

for the Hamiltonian

$$H(t) = \frac{p^2}{2m} + \mathcal{E}(t)x, \tag{2}$$

where $\mathcal{E}(t)$ is an arbitrary time-dependent real parameter and the mass of the particle m is constant. The solutions $x_c(t)$ and $p_c(t)$ for the classical equation of motion are given as

$$x_c(t) = x_0 + \frac{1}{m}\int_0^t p_c(t')dt', \tag{3}$$

$$p_c(t) = p_0 - \int_0^t \mathcal{E}(t')dt'. \tag{4}$$

The initial conditions x_0, p_0 correspond to the case where $\mathcal{E}(t) = 0$, i.e. to a free particle initial conditions.

In what follows, we briefly discuss the results of free particle, $\mathcal{E}(t) = 0$. The wave function $\psi_k^{free}(x,t)$ for the one-dimensional free particle with mass m evolves in position space according to the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi_k^{free}(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_k^{free}(x,t), \tag{5}$$

whose solution is described by a wave function corresponding to a plane wave:

$$\psi_k^{free}(x,t) = \exp\left[-\frac{i}{\hbar}E_k t\right]\frac{e^{ikx}}{\sqrt{2\pi}}, \tag{6}$$

where $E_k = \hbar^2k^2/2m$.

The general position- space solution $\Psi^{free}(x,t)$ to the time-dependent Schrödinger equation can be written, of course, as a linear combination of separable solutions ψ_k^{free}

$$\Psi^{free}(x,t) = \int_{-\infty}^{+\infty} g(k)\psi_k^{free}(x,t)dk, \tag{7}$$

where $g(k)$ is an arbitrary weight function for k. This wave packet describes a propagation of a free particle; its group velocity is identified as the velocity of the corresponding classical particle. This suggests we should view the "center" of the wave packet as travelling like a classical particle that obeys the laws of classical mechanics:

$$x_c^{free}(t) = x_0 + \frac{1}{m}p_0 t, \tag{8}$$

$$p_c^{free}(t) = p_0. \tag{9}$$

Since the corpuscular features (energy and momentum) of a particle are connected to its wave characteristics (wave frequency ω and wave number k) by the relations $E_k = \hbar\omega$ and $p = \hbar k$, we see how the wave packet concept offers a clear connection between the classical description of a particle and its quantum mechanical description.

After this brief discussion of the free particle results, let us now return to our main task of calculating the wave function of a particle in a time-dependent linear potential. For this, we establish a correspondence between the particle's velocity and the group velocity of the wave packets. This can be inferred immediately from (4): it turns out that the classical momentum for a particle in a time-dependent linear potential is obtained from that of a free particle by the substitution $p \to p^{free} - \int_0^t \mathcal{E}(t')dt'$, which suggest, from the de Broglie hypothesis, that the wave function for a particle in a time-dependent linear potential is obtained from that of a free particle by the substitution $k \to k - \frac{1}{\hbar}\int_0^t \mathcal{E}(t')dt'$. After performing this shift of k in (6)

$$\psi_k(x,t) = \psi^{free}_{k+\frac{C(t)}{\hbar}}(x,t) = \exp\left[-\frac{i}{\hbar}\int_0^t E_{k+\frac{C(t)}{\hbar}}\,dt\right]\frac{e^{i(k+\frac{C(t)}{\hbar})x}}{\sqrt{2\pi}} \tag{10}$$

and using the identities

$$\begin{aligned} E_{k+\frac{C(t)}{\hbar}} &= \hbar^2\left[k^2 + \frac{C(t)^2}{\hbar^2} + 2k\frac{C(t)}{\hbar}\right], \\ C(t) &= -\int_0^t \varepsilon(t')dt' = p_c(t) - p_c^{free}(t), \\ C_1(t) &= \int_0^t \frac{C(t')}{m}dt' = x_c(t) - x_c^{free}(t), \end{aligned} \tag{11}$$

we obtain the general expression of the wave function

$$\begin{aligned} \psi_k(x,t) = \frac{1}{\sqrt{2\pi}}\exp\left[\frac{-i\hbar k^2}{2m}t\right]\exp\left[-\frac{i}{2\hbar m}\int_0^t [C(t')]^2dt'\right]\exp[-iC_1(t)k] \\ \times\exp\left[\frac{i}{\hbar}C(t)x\right]\exp[ikx] \end{aligned}$$

$$= \exp\left[-\frac{i}{2\hbar m}\int_0^t [p_c(t') - p_c^{free}(t')]^2 dt'\right] \exp\left[\frac{i}{\hbar}\left\{p_c(t) - p_c^{free}(t)\right\}x\right]$$
$$\times\, \psi_k^{free}\left(x - \left\{x_c(t) - x_c^{free}(t)\right\}\right). \tag{12}$$

It is easy to check that this $\psi_k(x,t)$ satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi_k(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \mathcal{E}(t)x\right]\psi_k(x,t). \tag{13}$$

This is an ordinary one dimensional Schrödinger equation of a particle in a time-dependent linear potential.

At this point, any suitable choice of $g(k)$ yields conventional solutions as the Airy functions or a Gaussian wave packet.

(i) *Airy wave packets.* Putting $g(k) = \frac{\hbar^{2/3}}{\sqrt{2\pi}B}e^{i\hbar^2k^3/3B^3}$, making a variable change $k \to k + \frac{B^3\hbar}{2m}t$, where B is an arbitrary constant, using the representation of the Airy function

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ik^3/3}e^{ikx}dk, \tag{14}$$

and after integrating (7), one will obtain a nonspreading function, i.e. the Airy solution

$$\Psi(x,t) = \exp\left[-\frac{i}{2\hbar m}\int_0^t \left[p_c(t') - p_c^{free}(t')\right]^2 dt'\right]\exp\left[\frac{i}{\hbar}\left\{p_c(t) - p_c^{free}(t)\right\}x\right]$$
$$\times\, \Psi^{free}\left(x - \left\{x_c(t) - x_c^{free}(t)\right\}, t\right)$$
$$= \exp\left[-\frac{i}{2\hbar m}\int_0^t \left[p_c(t') - p_c^{free}(t')\right]^2 dt'\right]\exp\left[\frac{i}{\hbar}\left\{p_c(t) - p_c^{free}(t)\right\}x\right]$$
$$\times \exp\left[i\frac{B^3t}{2m\hbar}\left(x - \frac{B^3t^2}{6m^2}\right)\right]Ai\left(\frac{B}{\hbar^{2/3}}\left(x - \left\{x_c(t) - x_c^{free}(t)\right\} - \frac{B^3t^2}{4m^2}\right)\right). \tag{15}$$

We see that $\Psi^{free}(x,t)$ corresponds to the wave packet of [3, 4, 13, 15, 21, 22] which propagates in free space without distortion and with constant acceleration.

(ii) *Gaussian wave packets.* The standard Gaussian momentum-space distribution, which gives arbitrary initial momentum p_0 and position x_0 values, can be written in the form

$$g(k) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \exp(-a(\hbar k - p_0)^2)\exp(-ikx_0), \tag{16}$$

where a is a positive real constant. The explicit form of the position-space wave function is given by the Gaussian integral

$$\Psi(x,t) = \sqrt{\frac{\sqrt{a}}{\sqrt{2\pi}}}\exp\left[-\frac{i}{2\hbar m}\int_0^t \left[p_c(t') - p_c^{free}(t')\right]^2 dt'\right]\exp\left[\frac{i}{\hbar}\left\{p_c(t) - p_c^{free}(t)\right\}x\right]$$
$$\times \exp\left[-\frac{i}{\hbar}\left\{x_c(t) - x_c^{free}(t)\right\}p\right]$$
$$\times \int_{-\infty}^{+\infty} e^{ik(x-x_0)}\exp\left(-a(\hbar k - p_0)^2\right)\exp\left[-\frac{i\hbar k^2}{2m}t\right]dk, \tag{17}$$

which can be evaluated in closed form (using the change of variables $k \to \hbar k - p_0$ and standard integrals) to obtain a Gaussian wave packets solution for a particle in a time-dependent linear potential

$$\Psi(x,t) = \sqrt{\frac{\sqrt{a}}{\sqrt{2\pi}(a + \frac{i\hbar}{2m}t)}} \exp\left[-\frac{i}{2\hbar m}\int_0^t \left[p_c(t') - p_c^{free}(t')\right]^2 dt'\right] \times \exp\left[\frac{i}{\hbar}\left\{p_c(t) - p_c^{free}(t)\right\}x\right] \exp\left[-\frac{(x - x_c(t))^2}{4(a + \frac{i\hbar}{2m}t)}\right]. \tag{18}$$

It is crucial to note that the physically acceptable Gaussian solution (18) was obtained without any constraint imposed in the wave packet as was done in Luan et al. paper [16] (26).

The time-dependent expectation values of position and momentum are

$$\langle x\rangle_t = x_c(t), \qquad \langle p\rangle_t = p_c(t), \qquad \Delta x_t = \sqrt{\frac{(a^2 + \frac{\hbar^2}{4m^2}t^2)}{a}}, \qquad \Delta p = \frac{\hbar}{2}\sqrt{\frac{1}{a}} \tag{19}$$

all of which are familiar results and lead to the uncertainty relation

$$\Delta p \Delta x_t = \frac{\hbar}{2}\frac{\sqrt{(a^2 + \frac{\hbar^2}{4m^2}t^2)}}{a} \geq \frac{\hbar}{2}. \tag{20}$$

The equality holds at $t = 0$ and the Gaussian wave packet is called the minimum uncertainty wave packet.

We can rewrite (18) as follows

$$\Psi(x,t) = \sqrt{\frac{\hbar(a - \frac{i\hbar}{2m}t)}{\sqrt{2\pi a}\Delta x_t^2}} \exp\left[-\frac{i}{2\hbar m}\int_0^t \left[p_c(t') - p_c^{free}(t')\right]^2 dt'\right] \exp\left[\frac{i}{\hbar}\left\{p_c(t) - p_c^{free}(t)\right\}x\right] \times \exp\left\{\frac{-(a - \frac{i\hbar}{2m}t)[x - x_c(t)]^2}{4a\Delta x_t^2}\right\}. \tag{21}$$

Moreover, the time-dependent probability density associated with this Gaussian wave packet is Gaussian for all times

$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}\Delta x_t} \exp -\left\{\frac{(x - x_c(t))^2}{2\Delta x_t^2}\right\}, \tag{22}$$

we see that Δx_t represents the width of the wave packet at time t. It is also readily verified that the time-dependent probability density is conserved and is given by

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1. \tag{23}$$

Equations (21) and (22) describe a Gaussian wave packet that is centered at $\langle x\rangle = x_c(t)$ whose width Δx_t varies with time. So, during time t, the packet's center has moved from $x = 0$ to $x = x_c(t)$ and its width has expanded from $\Delta x_0 = \sqrt{a}$ to $\Delta x_t = \Delta x_0\sqrt{1 + (\frac{\hbar t}{2ma})^2}$. The wave packet therefore undergoes a distortion; although it remains Gaussian, its width broadens with time whereas its height, $\frac{1}{\sqrt{2\pi}\Delta x_t}$, decreases with time. Further, it should be noted that the width of the Gaussian packet does not depend on the external force $\mathcal{E}(t)$. Thus the shape of the wave packet is not changed by the external excitation. This means that the external force $\mathcal{E}(t)$ acts uniformly in the wave packet.

Before concluding let us make a few remarks about the wave function (12) can be rewritten as follow

$$\psi_k(x;t) = D\big(x_c(t), x_c^{free}(t)\big)\psi_k^{free}(x,t), \tag{24}$$

where the unitary operator

$$D\big(x_c(t), x_c^{free}(t)\big) = \exp\left[-\frac{i}{2\hbar m}\int_0^t \left[p_c(t') - p_c^{free}(t')\right]^2 dt'\right]\exp\left[\frac{i}{\hbar}\left\{p_c(t) - p_c^{free}(t)\right\}x\right] \\ \times \exp\left[-\frac{i}{\hbar}\left\{x_c(t) - x_c^{free}(t)\right\}p\right] \tag{25}$$

is the unitary displacement operator closely related to the one which is responsible for the existence of coherent states for a harmonic oscillator. For the case $x_0 = p_0 = 0$, this unitary displacement operator (25) coincide with that found by Song [22]. The wave function translates, with $\{x_c(t) - x_c^{free}(t)\}$, with no change in shape, just a change in phase. It is clear that the evolution of $\psi_k(x,t)$ can be obtained from that of $\psi_k^{free}(x,t)$ by applying a time unitary operator $D(x_c(t), x_c^{free}(t))$.

The unitary relation between a free-particle system and the corresponding linear system closely resembles the one for (generalized) harmonic oscillators, in that the probability distribution of the unitarily transformed wave function moves globally, according to the classical solution, from the distribution of the original wave function, while the shapes of the two distributions are the same [22]. The shapes could evolve under the time evolution, as in the generalized coherent states.

We note that, the Airy wave packet $\Psi(x,t)$ (15) solution for a particle in time-dependent linear potential is a displaced Airy wave packets resulting from the application of the displacement operator onto the freely Airy wave packets $\Psi^{free}(x,t)$. This leaves $\Psi(x,t)$ in the form of an Airy packet with a probability or number-density packet $|\Psi^{free}(x - \{x_c(t) - x_c^{free}(t)\}, t)|^2$ which retains its shape. It explains also why the Airy function remains nonspreading even with a time-dependent homogeneous potential. And the Gaussian wave packet solution for a particle in time-dependent linear potential $\Psi(x,t)$ (18) is a displaced Gaussian wave packets resulting from the application of the displacement operator onto the freely Gaussian wave packets $\Psi^{free}(x,t)$.

Finally we may observe that, to find the relation between the two systems i.e. the system of a free particle and the system of a particle in a linear potential, we start from the system of a free particle and show that the system of a particle in a linear potential is obtained by making the substitution $k \to k - \frac{1}{\hbar}\int_0^t \mathcal{E}(t')dt'$. This has been done by the use of the de Broglie hypothesis. The substitution of $k \to k - \frac{1}{\hbar}\int_0^t \mathcal{E}(t')dt'$ is similar to the application of the unitary transformation operator or "displacement operator" $D(x_c(t), x_c^{free}(t))$, on the wave function ψ_k^{free} (or a wave packet Ψ^{free}) for the one-dimensional free particle, as found by Song [22].

References

1. Feynman, R., Hibbs, A.: Quantum Mechanics and Path Integrals. McGraw-Hill, New York (1965)
2. Landau, L.D., Lifshitz, E.M.: Quantum Mechanics. Pergamon, Oxford (1977)

3. Cocke, S., Reichl, L.E.: Phys. Rev. A **53**, 1746 (1996)
4. Cocke, S., Reichl, L.E.: Phys. Rev. A **52**, 4515 (1995)
5. Kohn, W., Mattsson, A.E.: Phys. Rev. Lett. **81**, 3487 (1998)
6. Vitos, L., Johansson, B., Kollar, J., Skriver, H.L.: Phys. Rev. A **61**, 052511 (2000)
7. Vitos, L., Johansson, B., Kollar, J., Skriver, H.L.: Phys. Rev. B **62**, 10046 (2000)
8. Chung, M.S., Miskovsky, N.M., Cutler, P.H., Kumar N, N.: Appl. Phys. Lett. **76**, 1143 (2000)
9. Allen, S.S., Richadson, S.L.: J. Appl. Phys. **79**, 886 (1996)
10. Berry, M.V., Balazs, N.L.: Am. J. Phys. **47**, 264 (1979)
11. Rau, A., Unnikrishnan, K.: Phys. Lett. A **222**, 304 (1999)
12. Guedes, I.: Phys. Rev. A **63**, 034102 (2001)
13. Feng, M.: Phys. Rev. A **64**, 034101 (2001)
14. Bauer, J.: Phys. Rev. A **65**, 036101 (2002)
15. Bekkar, H., Benamira, F., Maamache, M.: Phys. Rev. A **68**, 016101 (2003)
16. Luan, P.-G., Tang, C.-S.: Phys. Rev. A **71**, 014101 (2005)
17. Dunkel, J., Trigger, S.A.: Phys. Rev. A **71**, 052102 (2005)
18. Bowman, G.E.: J. Phys. A **39**, 157 (2006)
19. Kim, S.P.: Korean J. Phys. Soc. **44**, 464 (2006)
20. Lu, G., Hai, W., Cai, L.: Phys. Lett. A **357**, 181 (2006)
21. Lin, C.-L., Hsiung, T.-C., Huang, M.-J.: Europhys. Lett. **83**, 30002 (2008)
22. Song, D.-Y.: Europhys. Lett. **62**, 622 (2003)
23. Gloge, D., Marcuse, D.: J. Opt. Soc. Am. **59**, 1629 (1969)
24. Nienhuis, G., Allen, L.: Phys. Rev. A **48**, 656 (1993)
25. Hau, H.A.: Waves and Fields in Optoelectronics. Prentice-Hall, Englewood Cliffs (1984)
26. Siegman, A.E.: Lasers. Oxford Univ. Press, Oxford (1986)
27. Krache, L., Maamache, M., Saadi, Y., Beniaiche, A.: Chin. Phys. Lett. **26**, 070307 (2009)